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The Jordan structure of Lie and Kac–Moody algebras

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Abstract. In this paper we establish a precise relation between the structures of Lie and Jordan algebras by presenting a method of constructing one type of algebra from the other. The examples of the Lie algebras associated to simple Jordan algebras $M_m^{(n)}$ and Clifford algebras are discussed in detail. The generalization of such arguments to infinite-dimensional Lie algebras in terms of Fermi fields is also discussed.

1. Introduction

It has become clear in the past few years that a semi-direct product of Kac-Moody (KM) and Virasoro algebras plays a crucial role in underlying the algebraic structure of physical systems possessing conformal invariance in two dimensions [1]. In fact, KM algebras can be thought of as having a more fundamental structure since from the Sugawara construction [2], the Virasoro algebra can be constructed from KM generators. The Virasoro generators obtained in this manner depend upon the representation of the KM algebra appearing in the Sugawara form, and hence upon the value of its central element K (see [3] for a review).

It is a well known fact that the representations of KM algebras can be constructed in terms of quantum fields [3]. In particular, the quark model construction provides KM currents bilinear in Fermi fields, i.e.

$$T^a(z)=(i/2)\psi^\alpha(z)M^a_{\alpha\beta}\psi^\beta(z) \tag{1.1}$$

where M^a are real antisymmetric matrices constituting a finite-dimensional representation of a compact Lie algebra g. $\psi^\alpha(z)$, $\alpha=1,2,\ldots,\dim M$ are real independent Fermi fields defined on a two-dimensional spacetime parametrized by the complex variable z. The Laurent coefficients of (1.1) are the generators of the affine Kac–Moody algebra $\hat{g}$ with central element given by the Dynkin index of the finite-dimensional representation [4] X_M as

$$K\delta^{ab}=\tfrac{1}{2}X_M\varphi^2\delta^{ab}=-\tfrac{1}{2}\mathrm{Tr}(M^aM^b) \tag{1.2}$$

where φ is the highest root of g.

We have a particular interest in studying representations of KM algebras for which the quantity $X=2K/\varphi^2$, called the level, takes unit values. These are the representations admitting conformal embeddings [5]. It has been shown that for g classical, i.e. SO(n), SU(n) and SP(n), level 1 representations can be obtained with M^a in the defining representation of g [6]. For the exceptional Lie algebras, F_4, E_6, E_7 and E_8 however, there is no matrix representation yielding $X=1$.

Although there are no such matrix representations for the exceptional algebras (apart from G_2), level 1 representations still can be constructed bilinearly in Fermi fields using vertex operators [7, 8]. These fields, however, are no longer independent in the sense that their operator product expansion is given by [9]

$$\psi^{\alpha}(Z)\psi^{\beta}(\xi)=\frac{Z^{1/2}}{(Z-\xi)^{1/2}}\psi^{\gamma}(Z)+\text{regular terms} \tag{1.3a}$$

$|Z|>|\xi|$ and $\alpha\neq\beta\neq\gamma$. That contrasts with the operator product expansion for independent fields, i.e.

$$\psi^{\alpha}(Z)\psi^{\beta}(\xi)=\frac{Z}{Z-\xi}\delta^{\alpha\beta}+\text{regular terms} \tag{1.3b}$$

$|Z|>|\xi|$.

The object of this paper is to establish the existence of a more fundamental algebraic structure underlying the construction of Lie (Kac–Moody) algebras. This structure is based upon Jordan algebras which unifies (1.3a) and (1.3b) as a field-theoretic version of a Jordan product as we now explain.

A Jordan algebra [10, 11] $\mathcal{J}$ is an algebra endowed with a symmetric (commutative) product

$$a\circ b=b\circ a \qquad a,b\in\mathcal{J} \tag{1.4}$$

satisfying the Jordan identity or power associativity law

$$(a^2\circ b)\circ a=a^2\circ(b\circ a). \tag{1.5}$$

A simple example can be seen by the Clifford algebra with generators γ^{α}, $\alpha=1,2,\ldots,N$ and Jordan product given by the anticommutator

$$\gamma^{\alpha}\circ\gamma^{\beta}\equiv\tfrac{1}{2}\{\gamma^{\alpha},\gamma^{\beta}\}=\delta^{\alpha\beta}\mathbf{1}. \tag{1.6}$$

Notice the resemblance between (1.6) and the operator product expansion (1.3b).

Further examples of Jordan algebras are provided by $n\times n$ Hermitian matrices over the real ($\mathbb{R}$), complex ($\mathbb{C}$) and quaternionic numbers ($\mathbb{Q}$) denoted by $M_n^{(1)}$, $M_n^{(2)}$ and $M_n^{(4)}$ respectively. For $n=3$ there is an exceptional Jordan algebra $M_3^{(8)}$ consisting of 3×3 Hermitian matrices over the octonions ($\mathbb{O}$). In all these cases the Jordan product is given by one half of the anticommutator. When restricted to off-diagonal matrices, the Jordan product resembles the operator product expansion (1.3a) [9, 12–14].

The main point of this paper is to establish a precise relation between elements of a Jordan algebra and the generators of a Lie algebra. We shall present an alternative derivation of the Tits construction [11, 15–17] which we believe will be useful in understanding the role of Jordan algebras in the construction of representations of Kac–Moody and Virasoro algebras.

Let $\mathcal{J}$ and L be two vector spaces and let L be decomposable as

$$L=T^1+T^2+T^3+D \tag{1.7}$$

where each one of the three subspaces T^i $(i=1,2,3)$ are isomorphic to $\mathcal{J}$, i.e. we can associate an element $T^i(a)\in T^i$ to every $a\in\mathcal{J}$. We want to endow $\mathcal{J}$ with a Jordan algebra structure whenever L is a Lie algebra and vice versa. These two algebraic structures are related through the formulae

$$[T^i(a),T^j(b)]=\mathrm{i}\varepsilon^{ijk}T^k(a\circ b)+\delta^{ij}D_{a,b} \tag{1.8a}$$

$$[D_{a,b},T^i(c)]=T^i(\mathcal{D}_{ab}(c))=-T^i((a,c,b)) \tag{1.8b}$$

$$[D_{a,b},D_{c,d}]=D_{\mathcal{D}_{a,b}(c),d}+D_{c,\mathcal{D}_{a,b}(d)} \tag{1.8c}$$

where '$\circ$' and $[,]$ denote the product laws in $\mathcal{J}$ and L respectively, $(a, b, c) = (a \circ b) \circ c - a \circ (b \circ c)$ is the associator in $\mathcal{J}$ and $D_{a,b}$ are elements of D which generate a subalgebra isomorphic to the derivation algebra of $\mathcal{J}$. For finite-dimensional semi-simple Jordan algebras all derivations are inner and of the form $\mathscr{D}_{a,b}(c) = -(a, c, b)$ [11, 17]. They satisfy the commutation relations

$$[\mathscr{D}_{a,b}, \mathscr{D}_{c,d}] = \mathscr{D}_{\mathscr{D}_{a,b}(c),d} + \mathscr{D}_{c,\mathscr{D}_{a,b}(d)}. \tag{1.9}$$

As a consequence of the Jordan identity the generators of $\mathscr{D}$ have to satisfy [11, 17]

$$\mathscr{D}_{a\circ b,c} + \mathscr{D}_{c\circ a,b} + \mathscr{D}_{b\circ c,a} = 0. \tag{1.10}$$

We show that given any Jordan algebra one can construct a Lie algebra through (1.8). On the other hand, we give some sufficient conditions for a Lie algebra in order that its commutators can be written as (1.8).

In section 3 we discuss the Clifford algebra generated by gamma matrices and the Jordan product given by (1.6). We show that there is a sequence of orthogonal Lie algebras that can be constructed out of these gamma matrices. Conversely, we show that the orthogonal Lie algebras B_n and D_n may be decomposed according to section 2, exhibiting their Jordan structure.

In section 4 we discuss the construction of the Freudenthal magic square out of the Jordan algebras $M_3^{(1)}$, $M_3^{(2)}$, $M_3^{(4)}$ and $M_3^{(8)}$ and its relation with dependent fermions. In section 5 we present the example of the Poincaré algebra which is related to a non-semisimple Jordan algebra. Finally in section 6 we discuss the possible extensions of our arguments to Kac–Moody algebras.

2. Relationship between Jordan and Lie algebras

In this section we prove relations (1.8) in both directions. We first consider L to be a Lie algebra decomposed into

$$L = L_{-1} + L_0 + L_1 \tag{2.1}$$

with

$$[L_m, L_n] \subset L_{m+n} \qquad m, n = 0, \pm 1. \tag{2.2}$$

It then follows that L_1 and L_{-1} are Abelian. In general such decomposition is obtained by a U_1 generator Q

$$[Q, L_m] = mL_m \tag{2.3}$$

which does not necessarily belong to L. The decomposition (2.1) is used in the Jordan pair method [18, 19] in the construction of Jordan algebras out of Lie algebras. Here we also make a connection of this with the Tits construction. Suppose that L possess an involutive automorphism σ, $(\sigma^2 = 1)$ providing a one to one map between the Abelian subspaces L_1 and L_{-1}, i.e.

$$\sigma(L_1) = L_{-1}. \tag{2.4}$$

This involution also decomposes the subalgebra L_0 into even and odd subspaces, i.e.

$$L_0 = D + T^3 \qquad \sigma(D) = D \qquad \sigma(T^3) = -T^3. \tag{2.5}$$

In the cases where $Q \in L$, (2.3) and (2.4) imply $\sigma(Q) = -Q$ and therefore $Q \in T^3$.

We denote by T^1 and T^2 the odd and even combinations according to σ of L_1 and L_{-1} respectively, i.e.

$$T^1 = L_1 - \sigma(L_1) \qquad T^2 = L_1 + \sigma(L_1). \tag{2.6}$$

Hence the Lie algebra L is decomposed as in (1.7). Using (2.2) and the fact that σ is an automorphism one gets

$$\begin{aligned} &[T^i, T^j] \subset T^k \qquad (i \neq j \neq k) \qquad [T^i, T^i] \subset D \\ &[D, T^i] \subset T^i \qquad [D, D] \subset D. \end{aligned} \tag{2.7}$$

The graded structure (2.1) defines another automorphism in L, namely σ' defined by $\sigma'(L_m) = (-1)^m L_m$. Where the integers m are eigenvalues of Q, σ' is realized by $\sigma'(L) = \exp(\mathrm{i}\pi Q) L \exp(-\mathrm{i}\pi Q)$. The involutions σ and σ' commute, hence define a third involution, $\sigma'' = \sigma\sigma' = \sigma'\sigma$. The identity map together with the three involutions in fact constitute a $Z_2 \times Z_2$ group. Associated with these involutions one can construct a series of symmetric spaces [20].

Given the grading (2.1) the involution σ satisfying (2.4) is not unique. By varying σ, we vary the subalgebra D and the subspace T^3 defined in (2.5). We are interested in those cases where the involution is such that there exists an automorphism of order 3, τ ($\tau^3 = 1$) permuting the subspaces T^i, $i = 1, 2, 3$ and leaving D invariant, i.e.

$$\tau(T^i) = T^{i+1 \bmod 3} \qquad \tau(D) = D. \tag{2.8a}$$

The vector spaces T^1 and T^2 are already isomorphic due to (2.6) and the existence of τ implies an isomorphism among T^1, T^2 and T^3. At this point we introduce a vector space $\mathcal{J}$ with elements denoted by $a, b, c, \ldots$ isomorphic to L_1. We shall denote a representative of an element $a \in \mathcal{J}$ in L_1 by $L_1(a)$. From the considerations above, the three subspaces T^i, $i = 1, 2, 3$ are also isomorphic to $\mathcal{J}$. We take the representative of $a \in \mathcal{J}$ in T^1 to be $T^1(a) = (1-\sigma)L_1(a)$. The representatives in T^2 and T^3 are then chosen to be $T^2(a) = \tau(T^1(a))$ and $T^3(a) = \tau^2(T^1(a))$ respectively. Then, we can write (2.8a) as

$$\tau(T^i(a)) = T^{i+1 \bmod 3}(a). \tag{2.8b}$$

Let us introduce a product law in $\mathcal{J}$, denoted by $a \circ b$, as follows: take the Lie bracket between the representative of a in T^1 and the representative of b in T^2. As a consequence of (2.7), the result is an element in T^3 which we define to be the representative of the element $a \circ b$, i.e.

$$[T^1(a), T^2(b)] \equiv \mathrm{i} T^3(a \circ b) \tag{2.9}$$

where the factor i has been introduced for convenience. Due to τ, this product can also be defined as $[T^2(a), T^3(b)]$ or $[T^3(a), T^1(b)]$. The fact that L_1 and L_{-1} are Abelian implies that

$$[(1-\sigma)L_1(a), (1+\sigma)L_1(b)] = [(1-\sigma)L_1(b), (1+\sigma)L_1(a)]. \tag{2.10a}$$

We want the product defined in (2.9) to be symmetric, i.e.

$$[T^1(a), T^2(b)] = [T^1(b), T^2(a)] \tag{2.10b}$$

and so, from (2.10a) we need $T^2(a) = \lambda(1+\sigma)L_1(a)$ where λ is a constant. In other words, we need $\tau((1-\sigma)L_1(a)) = \lambda(1+\sigma)L_1(a)$. In appendix 2 we discuss the conditions τ has to fulfil in order to get the symmetry of the product.

We shall now prove that our product satisfies the Jordan identity and hence defines a Jordan algebra. From (2.9) and the Jacobi identity we find

$$T^3((a^2, b, a)) = i[[[T^3(a), T^1(a)], T^2(a)], T^3(b)]. \tag{2.11}$$

When calculating (2.11), we have to keep in mind that, by definition (2.9), the result of an operation must be brought (using τ) to one of the subspaces T^1 or T^2 before operating with it again. The quantity $M_{312}(a) = [[T^3(a), T^1(a)], T^2(a)]$ is an element of D and being so is invariant under τ. Using this fact we conclude that $M_{123} = M_{312} = M_{231}$. It then follows from the Jacobi identity that they vanish and so does (2.11). The vector space $\mathcal{J}$ endowed with the product defined by (2.9) is then a Jordan algebra.

According to (2.7) the Lie bracket between two elements of the same subspace T^i is an element of D. Due to the triality τ, this is independent of i. We then define

$$D_{a,b} \equiv [T^i(a), T^i(b)] \qquad i = 1, 2, 3. \tag{2.12}$$

Therefore using (2.9) and the triality τ we establish (1.8a). Now using the definition (2.12), (1.8a) and the Jacobi identity we find that $D_{a,b}$ satisfy (1.8b) and (1.8c) (when choosing a convenient value for i).

Except for the possibility of the existence of elements of D which are not of the form (2.12) these are all the commutation relations for L. In any case, the elements $D_{a,b}$ constitute a subalgebra of D which realizes the derivation algebra of the Jordan algebra $\mathcal{J}$. We have then shown that any Lie algebra possessing the grading (2.1)–(2.2) and automorphisms σ and τ defined in (2.4) and (2.8) respectively, has a Jordan structure in the sense of (1.8).

Conversely we can prove that relations (1.8) can be used to construct a Lie algebra out of a Jordan algebra. Let $\mathcal{J}$ be an arbitrary Jordan algebra and L be a vector space with a decomposition given by (1.7), where T^i, $i = 1, 2, 3$ are three vector spaces isomorphic to $\mathcal{J}$. The inverse construction is proved by showing that the product [,] defined by (1.8) satisfies the Jacobi identity. Using the fact that $T^i(0) = D_{0,a} = D_{a,0} = 0$ one can easily check that the Jacobians† $j(T^i(a), D_{b,c}, D_{d,e})$ and $j(D_{a,b}, D_{c,d}, D_{e,f})$ vanish as a consequence of (1.9). Similarly $j(T^i(a), T^j(b), D_{c,d})$ vanishes due to the fact $\mathcal{D}_{a,b}$ is a derivation, i.e. $\mathcal{D}_{a,b}(c \circ d) = \mathcal{D}_{a,b}(c) \circ d + c \circ \mathcal{D}_{a,b}(d)$. Finally one gets

$$j(T^i(a), T^j(b), T^k(c)) = \mathrm{i}\varepsilon^{ijk}(D_{a\circ b,c} + D_{c\circ a,b} + D_{b\circ c,a}). \tag{2.13}$$

Although the derivation algebra $\mathcal{D}$ satisfies (1.10), the elements $D_{a,b} \in D$ do not, necessarily. In fact, $\Delta(a, b, c) = D_{a\circ b,c} + D_{c\circ a,b} + D_{b\circ c,a}$ generate a central element in the sense that it commutes with $T^i(a)$ and $D_{a,b}$. Due to the Levi-Civita symbol on the RHS of (2.13), the subspace $D + T^i$ (fixed i) is a Lie algebra even when the quantity $\Delta(a, b, c)$ does not vanish. The subalgebra D is then homomorphic to the derivation algebra $\mathcal{D}$ and $\Delta(a, b, c)$ lies in the kernel of such homomorphism. The vector space L is a Lie algebra only if D and $\mathcal{D}$ are actually isomorphic. Since the subspace Δ generated by $\Delta(a, b, c)$ is an Abelian ideal of D, the factor algebra $\bar{D} = D/\Delta$ is isomorphic to $\mathcal{D}$. Consequently the vector space $\bar{L} = T^1 + T^2 + T^3 + \bar{D}$ is always a Lie algebra. The quantity $\Delta(a, b, c)$ is in fact a 3-cocycle.

In the cases where the Jordan algebra has an identity element the automorphism (2.8) is realized by

$$\tau(l) = g_1 g_2 l g_2^{-1} g_1^{-1} \qquad l \in L \tag{2.14}$$

† We denote $j(l_1, l_2, l_3) = [[l_1, l_2], l_3] + [[l_3, l_1], l_2] + [[l_2, l_3], l_1]$.

where $g_i = \exp(i\pi T^i(1)/2)$. In addition, the representatives of the identity generate an SU(2) subalgebra commuting with D, i.e.

$$[T^i(1), T^j(1)] = i\varepsilon^{ijk}T^k(1) \tag{2.15}$$

$$[D_{a,b}, T^i(1)] = 0 \tag{2.16}$$

since $(a, 1, b) = 0$ for any $a, b \in \mathscr{J}$. We can define a bilinear form on $\mathscr{J}$ by restricting the Killing form of L to the subspace T^i, i.e. $\langle a, b\rangle = \mathrm{Tr}(T^i(a)T^i(b))$, which, due to τ, is independent of i. Notice that any automorphism σ of the Jordan algebra induces, via (1.8) an automorphism $\tilde{\sigma}$ of L by $\tilde{\sigma}(T^i(a)) = T^i(\sigma(a))$, $\tilde{\sigma}(D_{a,b}) = D_{\sigma(a),\sigma(b)}$. The relation between Lie and Jordan algebras given by (1.8) is known as the Tits construction [11, 15] (more precisely the first construction, see [11] for details). The SU(2) subalgebra (2.15) is not necessarily the same as the one appearing in the Tits second construction as the derivation algebra of the quaternions.

3. Clifford algebras and orthogonal Lie algebras

The simplest and most familiar example of a simple Jordan algebra consists of generators γ^α, $\alpha = 1, \ldots, N$ of a Clifford algebra with product given by the anticommutator (1.6). In this case the relations (1.8) read

$$[T^i(\gamma^\alpha), T^j(\gamma^\beta)] = i\varepsilon^{ijk}\delta^{\alpha\beta}T^k(1) + \delta^{ij}D_{\alpha,\beta} \tag{3.1a}$$

$$[T^i(1), T^j(\gamma^\alpha)] = i\varepsilon^{ijk}T^k(\gamma^\alpha) \tag{3.1b}$$

$$[D_{\alpha,\beta}, T^i(\gamma^\rho)] = \delta^{\beta,\rho}T^i(\gamma^\alpha) - \delta^{\alpha,\rho}T^i(\gamma^\beta) \tag{3.1c}$$

$$[D_{\alpha,\beta}, T^i(1)] = 0 \tag{3.1d}$$

$$[D_{\alpha,\beta}, D_{\gamma,\delta}] = \delta^{\beta\gamma}D_{\alpha,\delta} - \delta^{\alpha\gamma}D_{\beta,\delta} + \delta^{\beta\delta}D_{\gamma,\alpha} - \delta^{\alpha\delta}D_{\gamma,\beta} \tag{3.1e}$$

where we have used $(\gamma^\alpha, \gamma^\rho, \gamma^\beta) = \delta^{\alpha\rho}\gamma^\beta - \delta^{\rho\beta}\gamma^\alpha$, and $\mathscr{D}_{\alpha\beta}(1) = 0$. The element $D_{\mathbf{1},\alpha}$ does not appear on the RHS of (3.1b) since we have assumed the ideal Δ has been divided off from D. According to the arguments of section 2, the vector space $L = T^1 + T^2 + T^3 + D$ endowed with the operations (3.1) is a Lie algebra.

The Lie algebra D is easily recognizable since (3.1e) defines the special orthogonal algebra SO(N). Notice that (3.1a) for $i = j$ resembles the construction of SO(N) generators bilinearly in gamma matrices. Denoting $D_{N+1,\alpha} = iT^1(\gamma^\alpha)$ we see that (3.1a) and (3.1c) for $i = j = 1$ are written in the same form as (3.1e). Therefore the subspace $D + T^1(\gamma^\alpha)$, $\alpha = 1, \ldots, N$ is then the Lie algebra SO($N+1$). Adding the U_1 generator $T^1(1)$ we get the non-semisimple Lie algebra O($N+1$). Analogously denoting $D_{N+2,\alpha} = iT^2(\gamma^\alpha)$ and $D_{N+1,N+2} = iT^3(1)$ we can easily check that the subspace $D + T^1(\gamma^\alpha) + T^2(\gamma^\alpha) + T^3(1)$ constitute the Lie algebra SO($N+2$). Finally denoting $D_{N+3,\alpha} = iT^3(\gamma^\alpha)$, $D_{N+2,N+3} = iT^1(1)$ and $D_{N+1,N+3} = -iT^2(1)$ we find that the $L = \mathrm{SO}(N+3)$. Therefore the construction explained in section 2 provides, for the case of the Clifford algebra, a series of four orthogonal Lie algebras.

We now want to apply the inverse construction of section 2 to the orthogonal Lie algebras and obtain an underlying Clifford algebra (1.6). A practical procedure in decomposing a simple Lie algebra L into the form (2.1) is by taking the U_1 generator Q in (2.3) to have the form

$$Q = 2\lambda_a \cdot H/\alpha_a^2 \tag{3.2}$$

where α_a ($a=1,\ldots,\text{rank } L$) is a simple root of L and λ_a its corresponding fundamental weight. They satisfy $2\alpha_a\cdot\lambda_b/\alpha_a^2=\delta^{ab}$. The generators H_i ($i=1,\ldots,\text{rank } L$) constitute the Weyl–Cartan basis for the Cartan subalgebra of L and they obviously have zero grading WRT Q,

$$[Q, H_i]=0 \tag{3.3a}$$

$i=1,\ldots,\text{rank } L$. The step operators satisfy

$$[Q, E_\alpha]=(2\lambda_a\cdot\alpha/\alpha_a^2)E_\alpha=n_aE_\alpha \tag{3.3b}$$

where n_a are the integers in the expansion $\alpha=n_a\alpha_a$. The highest grade of Q is given by the coefficient m_a of α_a in the expansion of the highest root φ ($\varphi=m_a\alpha_a$). Therefore, Q has integer eigenvalues varying from $-m_a$ to m_a and hence it decomposes L into $(2m_a+1)$ subspaces. If we now want to decompose L as in (2.1), we have to choose λ_a such that $m_a=1$. For the orthogonal algebras SO$(2r)$ and SO$(2r+1)$ the highest root is given by $\alpha_1+2\alpha_2+\ldots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n$ and $\alpha_1+2\alpha_2+2\alpha_3+\ldots+2\alpha_n$ respectively. So, in order to construct the Clifford algebra (1.6), for both cases we have to choose Q in the direction of λ_1. The other two possibilities arising when $L=\text{SO}(2r)$ provide the Jordan algebras $\text{M}_N^{(4)}$, for $r=2N$. This example will be discussed in the next section.

The subalgebra L_0 is therefore generated by all Cartan subalgebra generators H_i together with the step operators E_α for roots α not containing α_1 in their expansion in terms of simple roots. The Dynkin diagram for L_0 is then obtained by deleting the point corresponding to α_1 in the Dynkin diagram of SO(N) ($N=2r+1$ or $2r$, [21], p 58). Consequently the orthogonal algebras are decomposed as

$$\text{SO}(N)=L_1+L_{-1}+\text{SO}(N-2)+U(1)_Q \tag{3.4}$$

where the $U(1)$ factor is generated by Q. The subspace L_1 (L_{-1}) is generated by all positive (negative) step operators for the roots containing α_1 in its expansion. In terms of an orthonormal basis e_i ($i=1,2,\ldots,r$) the roots of the orthogonal algebras are [20]

$$\text{roots of SO}(2r+1)=\{\pm e_i\pm e_j, \pm e_i, i,j=1,2,\ldots,r\} \tag{3.5a}$$

$$\text{roots of SO}(2r)=\{\pm e_i\pm e_j, i,j=1,2,\ldots,r\}. \tag{3.5b}$$

In both cases the fundamental weight λ_1 is equal to e_1, and since $\alpha_1^2=2$, we have [21]

$$Q=e_1H. \tag{3.6}$$

Therefore the subspace L_1 is given by†

$$L_1^{\text{SO}(2r+1)}=\{E_{e_1}, E_{e_1\pm e_l}, l=2,3,\ldots,r\} \tag{3.7a}$$

$$L_1^{\text{SO}(2r)}=\{E_{e_1\pm e_l}, l=2,3,\ldots,r\}. \tag{3.7b}$$

In both cases the subspace L_{-1} is generated by their corresponding negative root step operators.

The involution (2.4) in the case of the orthogonal algebras is an inner automorphism associated to the product of the Weyl reflections through the roots e_1+e_r and e_1-e_r. It is given by

$$\sigma(T)=\exp[\mathrm{i}\pi(S_2(e_1+e_r)+S_2(e_1-e_r))]T\exp[-\mathrm{i}\pi(S_2(e_1+e_r)+S_2(e_1-e_r))] \tag{3.8}$$

† Formally, these are generators of the complex algebras B_r and D_r and not of the compact real forms SO$(2r+1)$ and SO$(2r)$. The decomposition (3.4) is made, in fact, in the complexification of the orthogonal Lie algebras.

where $S_2(\alpha)$ is defined in (3.14). One can easily check that $\sigma(Q)=-Q$. It then follows from (2.3) and the fact that σ is an automorphism that $\sigma(L_1)$ has grade -1. Since σ is of order two, it satisfies (2.4).

Using the relations among the cocycles obtained in the appendix, one gets from (3.8)

$$\sigma(e_r H)=-e_r H \tag{3.9a}$$

$$\sigma(E_{e_1+e_r})=-E_{-(e_1+e_r)} \tag{3.9b}$$

$$\sigma(E_{e_1-e_r})=-E_{-(e_1-e_r)} \tag{3.9c}$$

$$\sigma(E_{e_1})=\varepsilon(e_1+e_r,-e_1)\varepsilon(e_1-e_r,e_r)E_{-e_1} \tag{3.9d}$$

$$\sigma(E_{e_r})=\varepsilon(e_1+e_r,-e_1)\varepsilon(e_1-e_r,e_r)E_{-e_r} \tag{3.9e}$$

$$\sigma(E_{e_1\pm e_l})=\varepsilon(e_1\pm e_l,-e_1-e_r)\varepsilon(-e_r\pm e_l,-e_1+e_r)E_{-e_1\pm e_l} \tag{3.9f}$$

$$\sigma(E_{e_r\pm e_l})=\varepsilon(e_r\pm e_l,-e_1-e_r)\varepsilon(e_1-e_r,-e_1\pm e_l)E_{-e_r\pm e_l} \tag{3.9g}$$

where $l=2,3,\ldots,r-1$. The generators $e_l H$, $E_{\pm e_l}$ and $E_{\pm e_l\pm e_m}$, $l,m=2,3,\ldots,r-1$, are invariant under σ.

According to (2.5), the subalgebra $L_0=\mathrm{SO}(N-2)\times U(1)$ split into $D+T_3$. For $N=2r$, the subspaces are given by

$$D^{\mathrm{SO}(2r)}=\{e_l H;\ E_{\pm e_l\pm e_m};\ E_{e_r\pm e_l}+\sigma(E_{e_r\pm e_l});\ l,m=2,3,\ldots,r-1\} \tag{3.10a}$$

$$T_3^{\mathrm{SO}(2r)}=\{Q,\ e_r H,\ E_{e_r\pm e_l}-\sigma(E_{e_r\pm e_l});\ l=2,3,\ldots,r-1\} \tag{3.10b}$$

and for the case $N=2r+1$ the subspaces are

$$D^{\mathrm{SO}(2r+1)}=\{D^{\mathrm{SO}(2r)},\ E_{e_r}+\sigma(E_{e_r}),\ E_{\pm e_l},\ l=2,3,\ldots,r-1\} \tag{3.11a}$$

$$T_3^{\mathrm{SO}(2r+1)}=\{T_3^{\mathrm{SO}(2r)},\ E_{e_r}-\sigma(E_{e_r})\}. \tag{3.11b}$$

Notice that $D^{\mathrm{SO}(2r)}$ is the algebra SO$(2r-3)$ since the generators $E_{e_r\pm e_l}+\sigma(E_{e_r\pm e_l})$ are the step operators for the short roots $\pm e_l$. On the other hand $D^{\mathrm{SO}(2r+1)}$ is the algebra SO$(2r-2)$ since $E_{e_r}+\sigma(E_{e_r})$ together with $e_l H$ generate the Cartan subalgebra, and suitable linear combinations of the generators $E_{\pm e_l}$ and $E_{e_r\pm e_l}+\sigma(E_{e_r\pm e_l})$ are the step operators for roots $\pm e_r\pm e_l$. In both cases the subspace T_3 has the same dimension as L_1 in (3.7).

Consider the inner automorphism defined as

$$\tau(T)=g(e_1+e_r)g(e_1-e_r)Tg^{-1}(e_1-e_r)g^{-1}(e_1+e_r) \tag{3.12}$$

where

$$g(\alpha)=\exp\left(-\mathrm{i}\frac{\pi}{2}S_1(\alpha)\right)\exp\left(-\mathrm{i}\frac{\pi}{2}S_2(\alpha)\right) \tag{3.13}$$

$$S_1(\alpha)=\frac{E_\alpha+E_{-\alpha}}{2}\qquad S_2(\alpha)=\frac{E_\alpha-E_{-\alpha}}{2\mathrm{i}}. \tag{3.14}$$

Using the relations in appendix 1, one can check that τ cyclically permutes the subspaces T_i ($i=1,2,3$), i.e.

$$\tau[(1-\sigma)E_{e_1}]=-\mathrm{i}(1+\sigma)E_{e_1} \tag{3.15a}$$

$$\tau[(1+\sigma)E_{e_1}]=\mathrm{i}\varepsilon(e_r, e_1-e_r)(1-\sigma)E_{e_r} \tag{3.15b}$$

$$\tau[(1-\sigma)E_{e_r}]=\varepsilon(e_r, e_1-e_r)(1-\sigma)E_{e_1} \tag{3.15c}$$

$$\tau[(1-\sigma)E_{e_1\pm e_r}]=-\mathrm{i}(1+\sigma)E_{e_1\pm e_r} \tag{3.16a}$$

$$\tau[(1+\sigma)E_{e_1\pm e_r}]=\mathrm{i}(e_1\pm e_r)\cdot H \tag{3.16b}$$

$$\tau[(e_1\pm e_r)\cdot H]=(1-\sigma)E_{e_1\pm e_r} \tag{3.16c}$$

$$\tau[(1-\sigma)E_{e_1\pm e_l}]=-\mathrm{i}(1+\sigma)E_{e_1\pm e_l} \tag{3.17a}$$

$$\tau[(1+\sigma)E_{e_1\pm e_l}]=\mathrm{i}\varepsilon(e_1\pm e_l, -e_1+e_r)(1-\sigma)E_{e_r\pm e_l} \tag{3.17b}$$

$$\tau[(1-\sigma)E_{e_r\pm e_l}]=\varepsilon(e_1\pm e_l, -e_1+e_r)(1-\sigma)E_{e_1\pm e_l} \tag{3.17c}$$

where $l=2,3,\ldots,r-1$ and τ leaves the elements of $D^{\mathrm{SO}(2r)}$ and $D^{\mathrm{SO}(2r+1)}$ invariant. Therefore this automorphism satisfies the requirements of section 2 and according to the arguments given there the Lie algebras SO($2r$) and SO($2r+1$) possess a Jordan structure in the sense of (1.8). The Jordan algebra obtained, from the product law (2.9) is indeed the Clifford algebra (1.6). The representatives of the identity and the γ-matrices in the subspace T_1 are given by

$$T^1(1)=\tfrac{1}{2}(1-\sigma)(E_{e_1+e_r}+E_{e_1-e_r}) \tag{3.18a}$$

$$T^1(\gamma_{2r-3})=\tfrac{1}{2}(1-\sigma)(E_{e_1+e_r}-E_{e_1-e_r}) \tag{3.18b}$$

$$T^1(\gamma_{2l-3)})=\tfrac{1}{2}(1-\sigma)(E_{e_1+e_l}-\eta_l E_{e_1-e_l}) \tag{3.18c}$$

$$T^1(\gamma_{2l-2})=\frac{-\mathrm{i}}{2}(1-\sigma)(E_{e_1+e_l}+\eta_l E_{e_1-e_l}) \tag{3.18d}$$

where $l=2,3,\ldots,r-1$, and η_l is the product of cocycles appearing in (3.9f), i.e.

$$\begin{aligned}\eta_l&=\varepsilon(e_1+e_l, -e_1-e_r)\varepsilon(-e_r+e_l, -e_1+e_r)\\&=\varepsilon(e_1-e_l, -e_1-e_r)\varepsilon(-e_r-e_l, -e_1+e_r).\end{aligned} \tag{3.19}$$

The equality between the product of these cocycles can be easily checked using the relations of appendix 1.

For the case of SO($2r+1$) there is an additional γ-matrix represented in T^1 by

$$T^1(\gamma_{2r-2})=(\varepsilon(e_1+e_r, -e_1)\varepsilon(e_r, e_1-e_r))^{1/2}(E_{e_1}-\sigma(E_{e_1}))/2. \tag{3.20}$$

The representatives in the subspaces T^2 and T^3 are obtained by applying τ and τ^2 respectively to the generators quoted above. That this choice of representatives does provide the algebra (1.6) can be easily verified by checking directly into (3.1) and using the relations displayed in appendix 1.

The step operators satisfy the Hermiticity condition $E^+_\alpha=E_{-\alpha}$. Therefore, from (3.9d) one observes that the square root of the product of cocycles in (3.20) guarantees the Hermiticity of $T^1(\gamma_{2r-2})$. The generators introduced in (3.18) are also Hermitian. Since the triality τ, defined in (3.12), is a unitary transformation it follows that the representatives in the subspace T^2 and T^3 are also Hermitian operators. Consequently the relations (1.8) for these examples are the commutation relations for the compact simple Lie algebra SO($2r$) and SO($2r+1$).

4. Jordan algebras and the Freudenthal magic square

In this section we shall apply the construction of section 2 to the Lie algebras appearing in the third line of the Freudenthal magic square [15–17], namely SP(3), SU(6), SO(12) and E_7 and show that they are related, via (1.8) to the simple Jordan algebras $M_3^{(1)}$, $M_3^{(2)}$, $M_3^{(4)}$ and $M_3^{(8)}$ respectively. These are the Jordan algebras of 3×3 Hermitian matrices over the real, complex, quaternionic and octonionic numbers with product given by half of the anticommutator. We will do that exploring the fact that SU(6), SO(12) and E_7 are obtained by matching $g_1 = SP(3)$ with $g_2 = SU(3)$, SP(3) and F_4 respectively. For more details about matching of Lie algebras see [8] and [9]. It is straightforward to generalize our results to show that the algebras SP(n), SU($2n$) and SO($4n$) are related to the Jordan algebras $M_n^{(1)}$, $M_n^{(2)}$ and $M_n^{(4)}$ respectively. In fact some of these results have already been discussed in a previous paper [14].

The roots of the Lie algebras L on the third line of the magic square can be written, using the matching and the notation of [8] as the long roots of $g_1 = SP(3)$ i.e., $\pm\sqrt{2}e_i$, $i = 1, 2$ and 3, the long roots Φ_L of g_2 together with $(\pm e_1 \pm e_2)/\sqrt{2}+\Omega_1$, $(\pm e_2 \pm e_3)/\sqrt{2}+\Omega_2$ and $(\pm e_1 \pm e_3)/\sqrt{2}+\Omega_3$ where Ω_i $i = 1$, 2 and 3 vanish for $L = SP(3)$ and correspond to the orbits of SU(3), SP(3) and F_4 for $L = SU(6)$, SO(12) and E_7 respectively. The set of long roots Φ_L and the unit length vectors in Ω_i are given according to the notation of [8] by

(a) for $g_2 = SU(3)$, Φ_L vanish and

$$\Omega_1 = \{\pm(e_4 - e_5)/\sqrt{2}\} \qquad \Omega_2 = \{\pm(e_5 - e_6)/\sqrt{2}\} \qquad \text{and} \qquad \Omega_3 = \{\pm(e_4 - e_6)/\sqrt{2}\} \tag{4.1a}$$

(b) for $g_2 = SP(3)$, $\Phi_L = \{\pm\sqrt{2}e_i, i = 4, 5 \text{ and } 6\}$ and

$$\Omega_1 = \{(\pm e_4 \pm e_5)/\sqrt{2}\} \qquad \Omega_2 = \{(\pm e_5 \pm e_6)/\sqrt{2}\} \qquad \text{and} \qquad \Omega_3 = \{(\pm e_4 \pm e_6)/\sqrt{2}\} \tag{4.1b}$$

(c) for $g_2 = F_4$, $\Phi_L = \{\pm e_i \pm e_j, i, j = 4, 5, 6 \text{ and } 7\}$ and

$$\Omega_1 = \{\pm e_i, i = 4, 5, 6, 7\} \qquad \Omega_2 = \{(\pm e_4 \pm e_5 \pm e_6 \pm e_7)/2\}_{\text{even}} \qquad \text{and} \qquad \Omega_3 = \{(\pm e_4 \pm e_5 \pm e_6 \pm e_7)/2\}_{\text{odd}} \tag{4.1c}$$

where Ω_2 and Ω_3 contain an even and odd number of minus signs respectively.

We shall now show that the algebras Sp(n), SU($2n$), SO($4n$) and E_7 can be decomposed according to the structure of section 2. As in section 3, the U_1 generator Q has the form (3.2), and the fundamental weights λ_a are chosen as follows: λ_n for Sp(n) and SU($2n$), λ_{2n-1} or λ_{2n} for SO($4n$) and λ_7 for E_7. The index convention correspond to the expansion of the highest root as $2\alpha_1+2\alpha_2+\ldots+2\alpha_{n-1}+\alpha_n$; $\alpha_1+\alpha_2+\ldots+\alpha_{2n}$; $\alpha_1+2\alpha_2+\ldots+2\alpha_{2n-2}+\alpha_{2n-1}+\alpha_{2n}$ and $2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$ for Sp(n), SU($2n$), SO($4n$) and E_7 respectively. Again the subalgebra L_0 is obtained by deleting the point in the Dynkin diagram of g corresponding to the chosen λ_a ([21], p 58). They are $L_0 = SU(n)\times U_1$, $SU(n)\times SU(n)\times U_1$, $SU(2n)\times U_1$, $E_6\times U_1$ according to $g = SP(n)$, SU($2n$), SO($4n$) and E_7 respectively. In fact the subalgebras L_0 above, except for the U_1 factor, are those appearing in the second line in the Freudenthal magic square. For simplicity, from now on we shall only consider $n = 3$. The second and third lines are constructed by matching the root system of $SU(3)/\sqrt{2}$ and SP(3) respectively with $SU(3)/\sqrt{2}$, SP(3) and F_4. The simple roots of SP(3) are those of

$SU(3)/\sqrt{2}$ together with $\sqrt{2}e_3$. Then the fundamental weight λ_a in (3.2) (which is orthogonal to all roots of L_0 and $\sqrt{2}e_3\cdot\lambda_a=1$) for all entries in the third line is $\lambda_a=(e_1+e_2+e_3)/\sqrt{2}$. We now define, following the notation of [8, 9], the Abelian subspaces L_1 and L_{-1} (see (2.3))

$$L_1=\{E^{(e_1+e_2)/\sqrt{2}+\Omega_1}, E^{(e_2+e_3)/\sqrt{2}+\Omega_2}, E^{(e_1+e_3)/\sqrt{2}+\Omega_3}, E^{\sqrt{2}e_i}, i=1,2,3\}$$

with $L_{-1}=(L_1)^+$ where Ω_1, Ω_2 and Ω_3 are defined in (4.1).

The vectors $(e_1+e_2)/\sqrt{2}$, $(e_2+e_3)/\sqrt{2}$ and $(e_1+e_3)/\sqrt{2}$ are in fact short roots of SP(3). The automorphism (2.4) mapping L_1 into L_{-1} can be constructed out of three Weyl reflections generated by the long roots of SP(3) namely: $\sqrt{2}e_1$, $\sqrt{2}e_2$ and $\sqrt{2}e_3$. It is defined as

$$\sigma(L)=\sigma_{\sqrt{2}e_1}(\sigma_{\sqrt{2}e_2}(\sigma_{\sqrt{2}e_1}(L))) \tag{4.2}$$

where $\sigma_x(L)=e^{i\pi S_2(x)}L\,e^{-i\pi S_2(x)}$ and $S_2(x)=(E^x-E^{-x})/2i$. It therefore follows, using the relations of appendix 1

$$\sigma(E^{\sqrt{2}e_i})=-E^{-\sqrt{2}e_i} \tag{4.3a}$$

$$\sigma(e_i\cdot H)=-e_iH \tag{4.3b}$$

$$\sigma(E^{(\xi e_i+\eta e_j)/\sqrt{2}+\Omega_{ij}})=\rho_{ij}^{(\xi,\eta)}E^{(-\xi e_i-\eta e_j)/\sqrt{2}+\Omega_{ij}} \tag{4.3c}$$

where

$$\rho_{ij}^{(\xi,\eta)}=\eta\xi\varepsilon\left(-\xi\sqrt{2}e_i,\frac{\xi e_i+\eta e_j}{\sqrt{2}}+\Omega_{ij}\right)\varepsilon\left(-\eta\sqrt{2}e_j,\frac{-\xi e_i+\eta e_j}{\sqrt{2}}+\Omega_{ij}\right)$$

and $\xi,\eta=\pm1$, $i,j,k=1,2,3$. We had denoted the orbits given in (4.1) as $\Omega_1=\Omega_{12}$, $\Omega_2=\Omega_{23}$ and $\Omega_3=\Omega_{13}$ (see paragraph preceding (4.1)). Since the Weyl reflections in (4.2) commute among themselves, we get $\sigma^2(L)=1$.

Apart from mapping L_1 into L_{-1} the automorphism σ defined in (4.2) also split L_0 into $D+T_3$ such that $\sigma(D)=D$ and $\sigma(T_3)=-T_3$. We get

$$T^3=\{(1-\sigma)E^{(e_i-e_j)/\sqrt{2}+\Omega_{ij}}, \sqrt{2}e_iH, i,j=1,2 \text{ and } 3\}$$

so that $\dim T_3=\dim L_1$. According to section 2 the identity elements belong to the centralizer of D. In fact there is a SU(2) subalgebra commuting with D generated by

$$T^1(1)=\left(\frac{E^{\sqrt{2}e_1}+E^{-\sqrt{2}e_1}}{2}\right)+\left(\frac{E^{\sqrt{2}e_2}+E^{-\sqrt{2}e_2}}{2}\right)+\left(\frac{E^{\sqrt{2}e_3}+E^{-\sqrt{2}e_3}}{2}\right) \tag{4.4a}$$

$$T^2(1)=\left(\frac{E^{\sqrt{2}e_1}-E^{-\sqrt{2}e_1}}{2i}\right)+\left(\frac{E^{\sqrt{2}e_2}-E^{-\sqrt{2}e_2}}{2i}\right)+\left(\frac{E^{\sqrt{2}e_3}-E^{-\sqrt{2}e_3}}{2i}\right) \tag{4.4b}$$

and

$$T^3(1)=\left(\frac{e_1+e_2+e_3}{\sqrt{2}}\right)H \tag{4.4c}$$

satisfying (2.15).

Let us now define τ by

$$\tau(T)=\exp\left(\frac{-i\pi}{2}T^1(1)\right)\exp\left(\frac{-i\pi}{2}T^2(1)\right)T\exp\left(\frac{i\pi}{2}T^2(1)\right)\exp\left(\frac{i\pi}{2}T^1(1)\right). \tag{4.5}$$

It is then straightforward to obtain

$$\tau((1-\sigma)E^{\sqrt{2}e_i}) = -\mathrm{i}(1+\sigma)E^{\sqrt{2}e_i} \tag{4.6a}$$

$$\tau((1+\sigma)E^{\sqrt{2}e_i}) = \mathrm{i}\sqrt{2}e_i H \tag{4.6b}$$

$$\tau(\sqrt{2}e_i H) = (1-\sigma)E^{\sqrt{2}e_i}. \tag{4.6c}$$

Also using the relations among the cocycles, given in appendix 1, and the fact that σ is an automorphism of L, it is tedious but straightforward to show that

$$\tau((1-\sigma)E^{(e_i+e_j)/\sqrt{2}+\Omega_{ij}}) = -\mathrm{i}(1+\sigma)E^{(e_i+e_j)/\sqrt{2}+\Omega_{ij}} \tag{4.7a}$$

$$\tau((1+\sigma)E^{(e_i+e_j)/\sqrt{2}+\Omega_{ij}}) = -\mathrm{i}\varepsilon\left(-\sqrt{2}e_j, \frac{e_i+e_j}{\sqrt{2}}+\Omega_{ij}\right)(1-\sigma)E^{(e_i-e_j)/\sqrt{2}+\Omega_{ij}} \tag{4.7b}$$

$$\tau((1-\sigma)E^{(e_i-e_j)/\sqrt{2}+\Omega_{ij}}) = -\varepsilon\left(-\sqrt{2}e_j, \frac{e_i+e_j}{\sqrt{2}}+\Omega_{ij}\right)(1-\sigma)E^{(e_i+e_j)/\sqrt{2}+\Omega_{ij}}. \tag{4.7c}$$

We denote the elements of the Jordan algebra $\mathrm{M}_3^{(n)}$ by

$$a_1 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \qquad a_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} \qquad \text{and} \qquad a_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

$$a_{12} = \begin{pmatrix} 0 & w & 0 \\ \bar{w} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad a_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & w \\ 0 & \bar{w} & 0 \end{pmatrix} \qquad a_{13} = \begin{pmatrix} 0 & 0 & w \\ 0 & 0 & 0 \\ \bar{w} & 0 & 0 \end{pmatrix} \tag{4.8}$$

where w and its conjugate $\bar{w}$ represent real, complex, quaternionic and octonionic numbers according to $n = 1$, 2, 4 and 8 respectively. We can check that the following choice of representatives

$$T^1(a_i) = (1-\sigma)E^{\sqrt{2}e_i} \qquad T^2(a_i) = \tau(T^1(a_i)) \qquad \text{and} \qquad T^3(a_i) = \tau^2(T^1(a_i)) \tag{4.9a}$$

and

$$T^1(a_{ij}) = (1-\sigma)E^{(e_i+e_j)/\sqrt{2}+\Omega_{ij}} \tag{4.9b}$$

$$T^2(a_{ij}) = \tau(T^1(a_{ij})) \qquad \text{and} \qquad T^3(a_{ij}) = \tau^2(T^1(a_{ij})) \tag{4.9c}$$

reproduces the multiplication table of the Jordan algebras (4.8) via (1.8).

Notice that the Jordan subalgebra generated by the elements a_i ($i = 1, 2, 3$) is related via (1.8) to the simply laced subalgebra $\mathrm{g}_L = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ common to Sp(3), SU(6), SO(12) and E_7. In all these cases g_L is generated by the step operators $E^{\pm\sqrt{2}e_i}$ together with the Cartan subalgebra generators $\sqrt{2}e_i \cdot H$, $i = 1, 2$ and 3.

5. The Poincaré algebra

Another example of our construction is provided by the Poincaré algebra. The commutations relations are

$$\begin{aligned} &[J_i, J_j] = \mathrm{i}\varepsilon_{ijk}J_k && [P_i, K_j] = \mathrm{i}\delta_{ij}P_0 \\ &[J_i, K_j] = \mathrm{i}\varepsilon_{ijk}K_k && [P_0, K_i] = \mathrm{i}P_i \\ &[K_i, K_j] = -\mathrm{i}\varepsilon_{ijk}J_k && [J_i, P_0] = 0 \\ &[J_i, P_j] = \mathrm{i}\varepsilon_{ijk}P_k && [P_\mu, P_\nu] = 0 \end{aligned} \tag{5.1}$$

where J_i, K_i and P_μ are the space rotations, boost and spacetime translations generators respectively ($i, j, k = 1, 2, 3$ and μ, $\nu = 0, 1, 2, 3$).

Taking the generator Q in (2.3) to be J_3, we decompose the Poincaré algebra as in (2.1) with $L_0 = \{J_3, K_3, P_3, P_0\}$, $L_1 = \{J_1 + iJ_2, K_1 + iK_2, P_1 + iP_2\}$ and $L_{-1} = L_1^+$. The automorphisms σ and τ are given by

$$\sigma(T) \equiv e^{i\pi J_2} T \, e^{-i\pi J_2} \tag{5.2a}$$

$$\tau(T) \equiv \exp(-i\pi J_1/2)\exp(-i\pi J_2/2) T \exp(i\pi J_2/2)\exp(i\pi J_1/2). \tag{5.2b}$$

One can easily check that these automorphisms fulfill the requirements of the construction explained in section 2. Therefore according to the arguments given there the Poincaré algebra is related to a Jordan algebra via equations (1.8). Such Jordan algebra, has an identity element $\mathbf{1}$ and is given by

$$\gamma_r \circ \gamma_s = g_{rs} \mathbf{1} \qquad r, s = 1, 2 \tag{5.3}$$

where

$$g_{rs} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.4}$$

One can check that this is indeed true by writing the relations (5.1) in the form (1.8) using the following choice of representatives of the Jordan elements

$$T_i(\mathbf{1}) \equiv J_i \qquad T_i(\gamma_1) \equiv K_i \qquad T_i(\gamma_2) \equiv P_i \tag{5.5a}$$

and

$$D_{\gamma_1, \gamma_2} \equiv -iP_0 \qquad D_{\mathbf{1}, \gamma_r} \equiv 0 \qquad r = 1, 2. \tag{5.5b}$$

The relations (1.8a) and (1.8b) can be checked, in this case, using the fact that all associators vanish, except $(\gamma_2, \gamma_1, \gamma_1) = \gamma_2$. The algebra (5.3) is not semisimple since γ_2 generates a solvable ideal. This is a consequence of the fact the Poincaré algebra is not semisimple. One easily sees that the Jordan subalgebra generated by $\mathbf{1}$ and γ_1 is semisimple and it is related, via (1.8), to the Lorentz algebra. Defining

$$\gamma_\pm \equiv \frac{\mathbf{1} \pm i\gamma_1}{2} \tag{5.6}$$

one gets

$$\gamma_\pm^2 = \gamma_\pm \qquad \gamma_+ \circ \gamma_- = 0 \qquad \gamma_\pm \circ \gamma_2 = \gamma_2/2 \qquad \gamma_2^2 = 0. \tag{5.7}$$

Therefore γ_2 and γ_+ (or γ_-) generate a non-semisimple subalgebra of (5.3) without identity. Such subalgebra is related, via (1.8), to the subalgebra of the Poincaré algebra constituted by the translations P_μ and the SU(2) generated by

$$N_i = \frac{J_i + iK_i}{2} \qquad i = 1, 2, 3. \tag{5.8}$$

This is an example where the equations (1.8) work for a Jordan algebra without identity.

6. Kac–Moody algebras and Fermi fields

We have related Jordan algebras to the construction of Lie algebras and shown that they possess a more fundamental structure in the sense that they can be thought of as building blocks for constructing certain Lie algebras. These Lie algebras, in turn, inherit the algebraic properties of the Jordan algebra. A similar picture can be found in constructing Kac–Moody algebras. The building blocks in this case are Fermion fields and their realization in terms of vertex operators is the most natural setting to be connected to the examples shown in sections 3 and 4 as we now discuss.

Let $Q^j(z)$ be the Fubini–Veneziano vector field

$$Q^j(z) = q^j - \mathrm{i}p^j \ln(z) + \mathrm{i} \sum_{n\neq 0} \alpha_n^j z^{-n}/n \tag{6.1}$$

where $[\alpha_m^i, \alpha_n^j] = m\delta^{ij}\delta_{m+n,0}$ and $[q^i, p^j] = \mathrm{i}\delta^{ij}$.

The vertex operator is given by

$$U^r(z) = z^{r^2/2} {:}\mathrm{e}^{\mathrm{i}r\cdot Q(z)}{:} \tag{6.2}$$

where the colons denote normal ordering in the sense that α_n^i, $n>0$ are moved to the right of those with $n<0$ and p^i to the right of q^i (see [3] for a review). The vertex operator (6.2) is a conformal field of weight $r^2/2$ with respect to the free bosonic stress energy momentum tensor for Q^i. Its Fermionic character arises whenever $r^2=1$ and a pair of real Fermi fields can be defined as

$$U^e(z)c_e = (\psi^e(z) - \mathrm{i}\psi^{-e}(z))/\sqrt{2} \tag{6.3}$$

where c_e are functions of p^i (Klein factors) necessary to provide the correct signs in the anticommutation relations for the ψ's. Information about dependence among Fermi fields is obtained from the singular part of the operator product expansion

$$U^e(z)U^{e'}(\xi) = z^{1/2}\xi^{1/2}(z-\xi)^{ee'} {:}\mathrm{e}^{\mathrm{i}eQ(z)+\mathrm{i}e'Q(\xi)}{:}. \tag{6.4}$$

A set of $2n$ independent real Fermi fields, ψ^i satisfying (1.3a) is associated to the orthogonal set of unit length vectors $\pm e_i$, $i=1,\ldots,n$, as

$$U^{\pm e_i}(z)c_{\pm e_i} = (\psi^{2i-1}(z) \mp \mathrm{i}\psi^{2i}(z))/\sqrt{2}. \tag{6.5}$$

These are considered fundamental building blocks for constructing Kac–Moody generators (with conformal weight 1, or $r^2=1$). Indeed for SO($2r$) they can be constructed bilinearly in Fermi fields (6.5), i.e. the step operators are given by

$$E^{\pm e_i \pm e_j} = U^{\pm e_i}(z)c_{\pm e_i}U^{\pm e_j}(z)c_{\pm e_j} \tag{6.6a}$$

and the Cartan subalgebra generators by

$$e_i H(z) = \mathrm{i}ze_i\, \mathrm{d}Q(z)/\mathrm{d}z = {}^\circ_\circ U^{e_i}(z)U^{e_i}(z){}^\circ_\circ \tag{6.6b}$$

where the open dots denote Fermionic normal ordering (see [3] for a review). Comparing (6.6) with the representatives of the gamma matrices in (3.18), we find that the combinations $U^{e_l} \pm U^{-e_l}$, $l=2,\ldots,r-1$ play the role, in the subspace T_1, of $2r-4$ gamma matrices while $U^{\pm e_r}$ characterizes γ_{2r-3} and the identity **1** in the same subspace with product given by the most singular part of the operator product expansion (1.3a). The other Fermi fields related to the even and odd combinations of U^{e_1} and U^{-e_1} characterize the subspaces T_1 and T_2. For SO($2r+1$) the extra short root step operators associated to $\pm e_i$ are constructed [8] as

$$E^{\pm e_i} = U^{\pm e_i}(z)c_{\pm e_i}\Psi(z) \qquad i=1,\ldots,r$$

where Ψ is an extra Fermi field independent of those defined in (6.5) and represent in T_1, the additional gamma matrix defined in (3.20).

Similarly to the description above, Fermionic vertex operators can be also constructed out of non-orthogonal unit length vectors. Consider for instance those displayed in Ω_i in (4.1). Within each orbit the operator product expansion (6.4) describes a set of independent Fermions (since the unit length vectors are either parallel, antiparallel or orthogonal). When considering vectors in distinct orbits, say Ω_i and Ω_j, $(i \neq j)$ the operator product expansion (6.4) acquires a square root branch cut as in (1.3*b*). This provides a realization of dependent Fermions whose properties are crucial in constructing level 1 representations of Kac–Moody algebras [8]. Again Kac–Moody generators are defined to be bilinears in such fields.

From (4.9) we find that in the subspace T_1 the representatives of the off-diagonal elements of the Jordan algebra $M_3^{(n)}$ decomposes into $(1-\sigma)\,U^{(e_i+e_j)/\sqrt{2}}$ which represents the position where the non-zero entries of a_{ij} lie, and $U^{\Omega_{ij}}$ describes the embedded division algebra. Indeed they do realize the algebra of the real, complex, quaternionic and octonionic numbers with product defined as in [9]. The generators corresponding to the diagonal part of $M_3^{(n)}$ can be obtained, in terms of Fermions, squaring the off-diagonal generators a_{ij} and using (1.8).

7. Conclusions

We have accomplished a relation between certain Lie and Jordan algebras. An attempt to extend such a relation to Kac–Moody algebras is proposed in terms of Fermi fields. Given a set of Fermi fields satisfying (1.3*a*) and (1.3*b*) we are now able to construct, following the identification discussed in section 6, certain Kac–Moody algebras. Further, the examples discussed in sections 3 and 4 provide from (1.8) an underlying Jordan structure. Moreover, (1.8) is universal whether Jordan or super Jordan algebras are concerned [22]. When extending to super Kac–Moody algebras it was shown in [22] that the supersymmetric counterpart of the independent Fermi fields are the so-called symplectic bosons used in [23].

We should also like to mention that the fourth line of the Freudenthal magic square actually correspond to a generalization of relations (1.8) in which instead of three we have seven subspaces $T_i, i=1,\ldots,7$ corresponding to the imaginary units of an octonion. The construction is based upon decomposing the exceptional Lie algebras F_4, E_6, E_7 and E_8 according to two U_1 generators.

We believe that our results may provide new insights towards constructing new representations of Kac–Moody algebras beyond those obtained from the quark model and the vertex operator construction [8].

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Appendix 1

The commutation relations for a simple Lie algebra L, in the Chevalley basis, are given by [21]

$$[H_a, H_b]=0 \qquad a, b=1,2,\ldots, \text{rank } L \tag{A1.1}$$

$$[H_a, E_\alpha]=(2\alpha_a\cdot\alpha/\alpha_a^2)E_\alpha \tag{A1.2}$$

$$[E_\alpha, E_\beta]=\begin{cases}(q+1)\varepsilon(\alpha,\beta)E_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \quad \text{(A1.3}a\text{)}\\ H_\alpha\equiv n_a H_a & \text{if } \alpha+\beta=0 \quad \text{(A1.3}b\text{)}\\ 0 & \text{otherwise} \quad \text{(A1.3}c\text{)}\end{cases}$$

where α_a are the simple root of L, n_a are the integers in the expansion $\alpha/\alpha^2=n_a\alpha_a/\alpha_a^2$, and q is the highest positive integer such that $\beta-q\alpha$ (or $\alpha-q\beta$) is a root. The cocycles $\varepsilon(\alpha,\beta)$ take the values ±1 and are obviously antisymmetric

$$\varepsilon(\alpha,\beta)=-\varepsilon(\beta,\alpha). \tag{A1.4}$$

The structure constants of L are completely determined from its root system except for the cocycles which are found by using Jacobi identities. If α, β and γ are roots adding up to zero, the Jacobi identity for their corresponding step operators yield

$$\varepsilon(\alpha,\beta)=\varepsilon(\beta,\gamma)=\varepsilon(\gamma,\alpha) \qquad \alpha+\beta+\gamma=0. \tag{A1.5}$$

Further relations are found by considering Jacobi identities of three step operators corresponding to roots adding up to a fourth root. Except for some cases occurring for the algebras C_n and F_4, the Jacobi identity for three step operators corresponding to non-proportional roots α, β and γ such that $\alpha+\beta+\gamma$ is a root, has only two terms. The reason is that the sum of a given pair of roots, let us say $\alpha+\gamma$, is not a root. In such cases the Jacobi identity implies

$$\varepsilon(\alpha,\beta)\varepsilon(\alpha+\beta,\gamma)=\varepsilon(\beta,\gamma)\varepsilon(\alpha,\beta+\gamma). \tag{A1.6}$$

For the algebras C_n and F_4 there can exist three short roots e, f and g such that the sum of any two of them and $e+f+g$ are roots. This happens when, let us say, $e\cdot f=0$ and $2e\cdot g/g^2=2e\cdot f/g^2=-1$. The three terms of the Jacobi identity for the corresponding step operators are non-vanishing and that implies

$$\varepsilon(e,f)\varepsilon(e+f,g)=\varepsilon(g,e)\varepsilon(f,g+e)=\varepsilon(f,g)\varepsilon(e,f+g). \tag{A1.7}$$

From the Jacobi identity for the step operators E_α, $E_{-\alpha}$ and $E_{\beta+n\alpha}$ with $1\leqslant n\leqslant p$, where p is the highest integer such that $\beta+p\alpha$ is a root, one obtains p relations involving the cocycles. By adding them up one obtains

$$\varepsilon(\alpha+\beta,-\alpha)\varepsilon(\alpha,\beta)=-1.$$

Then using (A1.5)

$$\varepsilon(\alpha,\beta)=-\varepsilon(-\alpha,-\beta). \tag{A1.8}$$

This relation implies the mapping $H_\alpha\to-H_\alpha$ and $E_\alpha\to-E_{-\alpha}$ is an automorphism of order two of the Lie algebra L.

Appendix 2

In this appendix we discuss the conditions the automorphism τ defined in (2.8) has to satisfy to guarantee the symmetry of the product law (2.9) (i.e. condition (2.10b)). As discussed there, τ has to obey $\tau((1-\sigma)L_1(a))=\lambda(1+\sigma)L_1(a)$ for some constant

λ. The mapping between T^1 and T^2 defined by τ can be described generically by a matrix M as $\tau((1-\sigma)L_1(a_m)) = M_{mn}(1+\sigma)L_1(a_n)$, where $L_1(a_m)$, $m = 1, 2, \ldots, \dim L_1$ is a basis for L_1. However one can always choose a basis where M is diagonal, i.e.

$$\tau((1-\sigma)L_1(a_m)) = \lambda_m(1+\sigma)L_1(a_m). \tag{A2.1}$$

Such diagonalization can be performed, in general, by using the irreducible components of the representation of D in L_1 as follows. The subspace L_1 (or L_{-1}) constitute a representation of the subalgebra D (see (2.2) and (2.5))

$$[D, L_1(a_m)] = R_{mn}(D)L_1(a_n). \tag{A2.2}$$

Since D is invariant under σ,

$$[D, (1\pm\sigma)L_1(a_m)] = R_{mn}(D)(1\pm\sigma)L_1(a_n). \tag{A2.3}$$

But since it is also invariant under τ,

$$\begin{aligned} \tau([D, (1-\sigma)L_1(a_m)]) &= M_{mn}[D, (1+\sigma)L_1(a_n)] \\ &= M_{mn}R_{nl}(D)(1+\sigma)L_1(a_l) \\ &= \tau(R_{mn}(D)(1-\sigma)L_1(a_n)) \\ &= R_{mn}(D)M_{nl}(1+\sigma)L_1(a_l). \end{aligned} \tag{A2.4}$$

Therefore M has to commute with all matrices of the representation R of D. From the Schur's lemma one gets that $M = \lambda 1$ if R is irreducible or M is block diagonal with each block being itself a multiple of an identity matrix if R is completely reducible. So the basis of the irreducible components of R diagonalizes M. Notice that when R is irreducible one gets that τ fulfills the condition for the product (2.9) to be symmetric. From (2.7) we have that $[T^i, T^i]$ is an element of D and so is invariant under τ. Then using (A2.1)

$$\begin{aligned} \tau([(1-\sigma)L_1(a_m)), (1-\sigma)L_1(a_n)] &= -(1+\sigma)[L_1(a_m), \sigma(L_1(a_n))] \\ &= \lambda_m\lambda_n[(1+\sigma)L_1(a_m), (1+\sigma)L_1(a_n)] \\ &= \lambda_m\lambda_n(1+\sigma)[L_1(a_m), \sigma(L_1(a_n))]. \end{aligned} \tag{A2.5}$$

So whenever $\Lambda_{m,n} \equiv [L_1(a_m), \sigma(L_1(a_n))] \neq 0$ one gets $\lambda_m\lambda_n = -1$. This imposes strong conditions on the possible eigenvalues of the matrix M. For instance, if $\Lambda_{1,2}$, $\Lambda_{1,3}$ and $\Lambda_{2,3}$ are non-zero one gets that $\lambda_1 = \lambda_2 = \lambda_3 = \pm i$. However there can exist cases where for a given m, $\Lambda_{m,n} = 0$ for all n. Then λ_m has apparently no relation with the other eigenvalues of M. If it turns out to be different from the others the product law (2.9) is not symmetric. In the examples of Clifford algebras and those appearing in the magic square discussed in this paper we have $(1+\sigma)[L_1(1), L_1(a)] = 0$ for all $a \in \mathcal{J}$. However the eigenvalue of M associated to $L_1(1)$ is equal to all others and consequently we do not have problems with the symmetry of the product (2.9). Further investigation is necessary to understand if that happens because of some unknown additional conditions or if the referred eigenvalue can be rescaled to be equal to all others.

References

[1] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 Infinite conformal symmetry in two-dimensional quantum field theory *Nucl. Phys.* B **241** 333–80

[2] Sugawara H 1968 A field theory of currents *Phys. Rev.* **170** 1659–62

[3] Goddard P and Olive D 1986 Kac–Moody and Virasoro algebras in relation to quantum physics *Int. J. Mod. Phys.* A **1** 303–414

[4] Dynkin E B 1957 Semi-simple subalgebras of semi-simple Lie algebras *Am. Math. Soc. Transl.* series 2 **6** 111–244; 1957 Maximal subgroups of the classical groups *Am. Math. Soc. Transl.* series 2 **6** 245–379

[5] Bais F A and Bouwknegt P G 1987 A classification of subgroup truncation of bosonic string *Nucl. Phys.* B **279** 561–70

[6] Arcuri R C, Gomes J F and Olive D 1987 Conformal subalgebras and symmetric spaces *Nucl. Phys.* B **285** 327–39

[7] Frenkel I B and Kac V G 1980 Basic representations of affine Lie algebras and dual resonance models *Invent. Math.* **62** 23–66

Segal G 1981 Unitary representations of some infinite dimensional groups *Commun. Math. Phys.* **80** 301–42

[8] Goddard P, Nahm W, Olive D and Schwimmer A 1986 Vertex operators for non simply-laced algebras *Commun. Math. Phys.* **107** 179–212

[9] Goddard P, Nahm W, Olive D, Ruegg H and Schwimmer A 1987 Fermions and octonions *Commun. Math. Phys.* **112** 385–408

[10] Jordan P, von Neumann J and Wigner E 1934 On an algebraic generalization of the quantum mechanical formalism *Ann. Math.* **35** 29–64

[11] Jacobson N 1968 *Structure and Representations of Jordan Algebra* (Am. Math. Soc. Colloq. Publ.); 1971 *Exceptional Lie Algebras* (New York: Dekker)

[12] Corrigan E and Hollowood T 1988 A string construction for a commutative non-associative algebra related to the exceptional Jordan algebra *Phys. Lett.* **203B** 47–51; 1989 The exceptional Jordan algebra and the superstring *Commun. Math. Phys.* **122** 393–410

[13] Gunaydin M and Hyun J 1988 Affine exceptional Jordan algebra and vertex operators *Phys. Lett.* **209B** 498–502

[14] Ferreira L A, Gomes J F and Zimerman A H 1988 Vertex operators and Jordan fields *Phys. Lett.* **214B** 367–70

[15] Tits J 1966 Algebres alternatives, algebres de Jordan et algebres de Lie exceptionnelles *Proc. K. Ned. Akad. Wet.* **A69** 223; 1953 Le plan projectif des octaves et les groupes de Lie excepcionnels *Acad. R. Belg. Bull. Cl Sci.* **V39** 309–29

[16] Freudenthal H 1964 Lie groups in the foundations of geometry *Adv. Math.* **1** 145–90

[17] Schafer R 1966 *An Introduction to Non Associative Algebras* (New York: Academic)

[18] Meyberg K 1970 Jordan triplesysteme und die Koecher konstruktion von Lie algebras *Math. Z.* **115** 58–78

[19] Loos O 1975 *Jordan Pairs Lecture Notes in Mathematics* vol 460 (Berlin: Springer)

[20] Helgason S 1978 *Differential Geometry, Lie Groups and Symmetric Spaces* (New York: Academic)

[21] Humphreys J 1972 *Introduction to Lie Algebras and Representation Theory* (Berlin: Springer)

[22] Ferreira L A, Gomes J F, Teotonio Sobrinho P and Zimerman A H 1990 *Phys. Lett.* **234B** 315–20

[23] Goddard P, Waterson G and Olive D 1987 *Commun. Math. Phys.* **112** 591